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Abstract
We explore the classical version of the mapping, due to Abanov and Wiegmann,
of Calogero–Sutherland hydrodynamics onto the Benjamin–Ono equation 'on
the double'. We illustrate the mapping by constructing the soliton solutions
to the hydrodynamic equations, and show how certain subtleties arise from
the need to include corrections to the naïve replacement of singular sums by
principal-part integrals.

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1. Introduction

The Calogero–Sutherland family of models [1, 2] consist of point particles moving on a line
or circle and interacting with a repulsive inverse-square potential. Both the classical and
quantum versions are completely integrable, and are the subject of an extensive literature [3].
The models have application to one-dimensional electron systems [4] and to the two-
dimensional quantum Hall effect [5].

It is possible to consider a hydrodynamic limit of the Calogero–Sutherland models in
which the distribution and velocity of the particles are described by continuous fields $\rho(x)$
and $v(x)$ respectively. The equations of motion of these fields possess solitary wave solutions,
and also periodic solutions that interpolate between small amplitude sound waves and large
amplitude trains of solitary waves [6, 7]. The quantum version of the hydrodynamics provides
an extension of the usual theory of bosonization of relativistic (i.e. linear dispersion) electron
systems to systems where band-curvature effects become important [8]. Because of the
singular nature of the Calogero–Sutherland interaction, the hydrodynamic limit is rather more
subtle than one might expect. Considerable insight into this limit has been provided by
Abanov and Wiegmann who have shown [9] that it is equivalent to a 'doubled' version of
Benjamin–Ono dynamics [10, 11]. The Benjamin–Ono equation, a member of an infinite
hierarchy of integrable partial differential equations, was originally introduced to describe

waves in stratified fluids. The connection has led to a number of novel predictions for the evolution of one-dimensional electron gases [12].

In this paper we will apply the tools of Abanov and Wiegmann [9] to recover the classical soliton solutions found in [6, 7] and, in doing so, illustrate the structure of the Calogero–Sutherland ↔ Benjamin–Ono mapping. A secondary aim is to compare the origin of the hydrodynamic-limit subtleties in the classical model with their origin in the quantum system. In the quantized model, they arise because of the need to convert operators that are Hermitian with respect to one of the two natural inner products on the Calogero–Sutherland Hilbert space into operators that are Hermitian with respect to the other [13]. In the classical model, the subtleties arise because we need to include corrections to the naïve limit of singular sums.

In section 2, we review the connection between the Calogo–Sutherland and Benjamin–Ono equations of motion. In section 3 we consider an appealing, but overly naïve, version of the hydrodynamic limit and reveal its failings. In section 4, we illustrate how these shortcomings are eliminated by including the first non-trivial order in an asymptotic expansion for the velocity field. In section 5, we consider the general mapping. The appendix contains proofs of some results used in the main text.

2. Calogero–Sutherland from the Benjamin–Ono pole ansatz

The classical Benjamin–Ono equation [10, 11] is the nonlinear and non-local partial differential equation

$$\dot{u} + u\partial_x u = \tfrac{1}{2}\lambda\left(\partial_{xx}^2 u\right)_{\mathrm{H}},\tag{1}$$

where f_{H} denotes the Hilbert transform of f:

$$f_{\mathrm{H}}(x) \stackrel{\mathrm{def}}{=} \frac{\mathrm{P}}{\pi}\int_{-\infty}^{\infty}\frac{1}{x-\xi}\,f(\xi)\,\mathrm{d}\xi.\tag{2}$$

If we introduce a Poisson bracket

$$\{u(x), u(x')\} = 2\lambda\pi\,\partial_x\delta(x-x')\tag{3}$$

and Hamiltonian

$$H_{\mathrm{BO}} = \frac{1}{2\lambda\pi}\int_{-\infty}^{\infty}\left\{\frac{1}{6}u^3 - \frac{\lambda}{4}u(u_x)_{\mathrm{H}}\right\}\,\mathrm{d}x,\tag{4}$$

then (1) can be written as $\dot{u}(x, t) = \{H_{\mathrm{BO}}, u(x, t)\}$. This Hamiltonian system possesses infinitely many Poisson-commuting integrals of motion [14].

Following [15], we seek solutions of (1) as a sum of poles

$$u(x, t) = \sum_{j=1}^{N}\frac{\mathrm{i}\lambda}{x - a_j(t)} - \sum_{j=1}^{M}\frac{\mathrm{i}\lambda}{x - b_j(t)}.\tag{5}$$

The poles at $a_j(t)$, $j = 1, \ldots, N$, lie below the real axis while the poles at $b_j(t)$, $j = 1, \ldots, M$, lie above it. In [15], the numbers of a_j and b_j poles were set equal and $b_j(t) = a_j^*(t)$. These conditions were imposed to ensure that $u(x, t)$ was real. We will not make these assumptions, so our field $u(x, t)$ is not necessarily real valued.

We insert the ansatz (5) into (1) and use

$$\left(\frac{1}{x-a}\right)_{\mathrm{H}} = \begin{cases} \dfrac{-\mathrm{i}}{x-a}, & \mathrm{Im}\,a < 0 \\[2mm] \dfrac{+\mathrm{i}}{x-a}, & \mathrm{Im}\,a > 0 \end{cases}\tag{6}$$

to find

$$\left\{\sum_{k=1}^{N}\frac{i\lambda\dot{a}_k}{(x-a_k)^2}-\sum_{k=1}^{M}\frac{i\lambda\dot{b}_k}{(x-b_k)^2}\right\}-\left\{\sum_{j=1}^{N}\frac{i\lambda}{x-a_j}-\sum_{j=1}^{M}\frac{i\lambda}{x-b_j}\right\}$$
$$\times\left\{\sum_{k=1}^{N}\frac{i\lambda}{(x-a_k)^2}-\sum_{k=1}^{M}\frac{i\lambda}{(x-b_k)^2}\right\}$$
$$-\lambda\left\{\sum_{j=1}^{N}\frac{\lambda}{(x-a_j)^3}+\sum_{j=1}^{M}\frac{\lambda}{(x-b_j)^3}\right\}=0. \tag{7}$$

All terms with $1/(x-a_i)^3$ and $1/(x-b_i)^3$ cancel directly. The remaining terms can be simplified by exploiting the identity

$$\frac{1}{(x-c_1)}\frac{1}{(x-c_2)^2}+\frac{1}{(x-c_2)}\frac{1}{(x-c_1)^2}=\frac{1}{(c_2-c_1)}\frac{1}{(x-c_2)^2}+\frac{1}{(c_1-c_2)}\frac{1}{(x-c_1)^2} \tag{8}$$

to rearrange them as a sum of $1/(x-a_i)^2$'s and $1/(x-b_i)^2$'s with x-independent coefficients. Now the set of $1/(x-a_i)^2$'s and $1/(x-b_i)^2$'s is linearly independent, and the vanishing of their individual coefficients requires[1]

$$i\dot{a}_j=\sum_{k;k\neq j}\frac{\lambda}{(a_k-a_j)}-\sum_{k=1}^{M}\frac{\lambda}{(b_k-a_j)}, \tag{9}$$

$$i\dot{b}_j=\sum_{k;k\neq j}\frac{\lambda}{(b_j-b_k)}-\sum_{k=1}^{N}\frac{\lambda}{(b_j-a_k)}. \tag{10}$$

We have found $N+M$ evolution equations for $N+M$ variables, and so the pole ansatz is internally consistent.

We now compute $\ddot{a}_j$ by differentiating (9), and then using equations (9) and (10) to eliminate $\dot{a}_i$ and $\dot{b}_i$'s respectively. After some labour involving repeated use of (8), we find that

$$\ddot{a}_j=\sum_{k;k\neq j}\frac{2\lambda^2}{(a_j-a_k)^3}. \tag{11}$$

Remarkably, $b_j(t)$ do not appear. The only role of b_j's in the a_i-pole dynamics is that the complex parameters $b_j(0)$ determine the (complex) initial velocities $\dot{a}_i(0)$ of the $a_i(t)$-poles. Once these initial conditions are established, the a_i-poles evolve autonomously according to (11).

The N equations (11) are a complex version of the Calogero model equations of motion. As for the real-a_i case, they can be derived from the many-body Lagrangian

$$L_{\text{Calogero}}=\frac{1}{2}\sum_{i=1}^{N}\dot{a}_i^2-\frac{1}{2}\sum_{i,j;i\neq j}\frac{\lambda^2}{(a_i-a_j)^2}. \tag{12}$$

---

[1] These equations are similar to, but not identical with, the Kelvin evolution of point vortices. The difference is that for a set of vortices of strength κ_i at point z_i, we have equations of the form

$$\dot{z}_i^*=\frac{1}{2\pi i}\sum_{j\neq i}\frac{\kappa_j}{z_i-z_j},$$

in which the complex conjugate of $\dot{z}_i$ is given by a pole sum involving only unconjugated z_i.

We may site our initial poles so that if we place a pole at z, then we also place one at $z + 2\pi n$ for any integer n. The resulting 2π periodicity will be preserved by the subsequent evolution. The sums

$$\lim_{M \to \infty} \left\{ \sum_{n=-M}^{M} \frac{1}{z + 2\pi n} \right\} = \frac{1}{2} \cot \left(\frac{z}{2} \right), \tag{13}$$

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z + 2\pi n)^2} = \frac{1}{4} \mathrm{cosec}^2 \left(\frac{z}{2} \right) \tag{14}$$

then allow us to write the evolution equations in the form

$$\mathrm{i}\dot{a}_j = \sum_{k; k \neq j} \frac{\lambda}{2} \cot \left(\frac{a_k - a_j}{2} \right) - \sum_{k=1}^{M} \frac{\lambda}{2} \cot \left(\frac{b_k - a_j}{2} \right), \tag{15}$$

$$\mathrm{i}\dot{b}_j = \sum_{k; k \neq j} \frac{\lambda}{2} \cot \left(\frac{b_j - b_k}{2} \right) - \sum_{k=1}^{N} \frac{\lambda}{2} \cot \left(\frac{b_j - a_k}{2} \right), \tag{16}$$

and

$$\ddot{a}_j = -\frac{\partial}{\partial a_j} \left\{ \frac{1}{4} \sum_{k; k \neq j} \frac{\lambda^2}{\sin^2(a_j - a_k)/2} \right\}. \tag{17}$$

If we regard a_i as angles, then (17) is the equation of motion arising from the Sutherland-model Lagrangian

$$L_{\text{Sutherland}} = \frac{1}{2} \sum_{i=1}^{N} \dot{a}_i^2 - \frac{1}{8} \sum_{i,j; i \neq j} \frac{\lambda^2}{\sin^2(a_i - a_j)/2} \tag{18}$$

for particles on a circle.

Chen *et al* [15] made $b_i = a_i^*$, as they wished $u(x, t)$ to be real. Being interested primarily in Calogero–Sutherland models, we instead desire that $a_i(t)$ be real. To arrange for this, we modify the definition of the Hilbert transform (2) appearing in (1). Following [9], we define the Γ-contour Hilbert transform of u to be

$$u_\Gamma(z) \stackrel{\text{def}}{=} \frac{\mathrm{P}}{\pi} \oint_\Gamma \frac{1}{z - \xi} f(\xi) \, \mathrm{d}\xi, \tag{19}$$

where Γ is a simple closed contour on which z lies. All that was needed to establish the a-pole autonomy was that $1/(x - a_i)$'s and $1/(x - b_i)$'s be eigenfunctions of the Hilbert transform with eigenvalues of opposite sign. Now if we take Γ to encircle the real axis in a clockwise sense (as shown in figure 1) then, for z on the contour,

$$\left(\frac{1}{z - a_i} \right)_\Gamma = \frac{-\mathrm{i}}{z - a_i}, \qquad \left(\frac{1}{z - b_i} \right)_\Gamma = \frac{+\mathrm{i}}{z - b_i}, \tag{20}$$

when the a_i-poles lie within Γ and the b_i-poles lie outside. We can therefore let $a_i(0)$ lie on the real axis and distribute the b-poles in the remainder of the complex plane in such a manner that the initial $\dot{a}_i$'s are real. Once the initial a_i's and $\dot{a}_i$'s are real, the Calogero evolution ensures that $a_i(t)$ remain on the real axis. Thus, a complex Benjamin–Ono field $u(z, t)$ obeying

$$\dot{u} + u\partial_z u = \tfrac{1}{2}\lambda \left(\partial_{zz}^2 u \right)_\Gamma \tag{21}$$

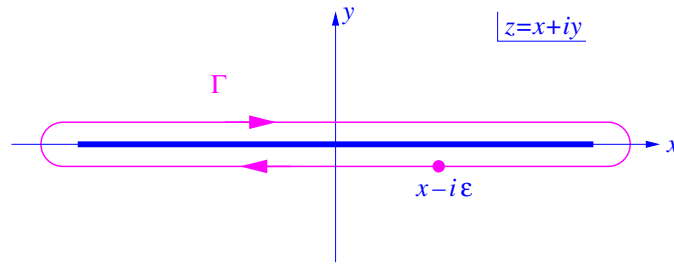on Γ can provide real-axis Calogero–Sutherland dynamics.

Figure 1. The contour Γ surrounds the part of the real axis on which the a_i-poles are found.

3. Shepherd poles and Calogero density-wave solitons

In this section, we will apply the pole ansatz to obtain solitary wave solutions to the hydrodynamic limit of the real-axis Calogero equation. In this limit we let the number N of Calogero particles become infinite, while keeping their density finite. We then replace the individual $a_i(t)$ by a smooth particle-number density $\rho(x, t)$ and the individual velocities $\dot{a}_i(t)$ by a smooth velocity field $v(x)$. We assume that ρ and v vary slowly on the scale of the inter-particle spacing. We are now naturally tempted to approximate the discrete a-pole sums in the evolution equations:

$$
\begin{aligned}
\mathrm{i}\dot{a}_j &= \sum_{k;k\neq j} \frac{\lambda}{a_k - a_j} - \sum_k \frac{\lambda}{b_k - a_j}, \\
\mathrm{i}\dot{b}_j &= \sum_{k;k\neq j} \frac{\lambda}{b_j - b_k} - \sum_k \frac{\lambda}{b_j - a_k}
\end{aligned}
\tag{22}
$$

by integrals to get

$$
\begin{aligned}
\mathrm{i}\dot{a}(x, t) &= \mathrm{P}\int_{-\infty}^{\infty} \frac{\lambda}{\xi - x}\rho(\xi, t)\,\mathrm{d}\xi - \sum_k \frac{\lambda}{b_k - x}, \\
\mathrm{i}\dot{b}_j(t) &= \sum_{k;k\neq j} \frac{\lambda}{b_j - b_k} - \int_{-\infty}^{\infty} \frac{\lambda}{b_j - \xi}\rho(\xi, t)\,\mathrm{d}\xi.
\end{aligned}
\tag{23}
$$

Here, $\dot{a}(x, t) \equiv v(x, t)$ is the velocity of the pole at x. We begin by exploring the consequences of this approximation to the discrete sum. We will see that it is not quite consistent, and a small but significant correction is needed.

Consider an initial density fluctuation

$$
\rho(x, 0) = \rho_0 + \rho_1(x), \qquad \text{where} \quad \rho_1(x) = \left(\frac{A}{\pi}\right)\frac{1}{x^2 + A^2}.
\tag{24}
$$

Because

$$
\int_{-\infty}^{\infty} \left(\frac{A}{\pi}\right)\frac{1}{\xi^2 + A^2}\,\mathrm{d}\xi = 1,
\tag{25}
$$

the Lorentzian distribution $\rho_1(x)$ corresponds to a local excess of one particle near $x = 0$.

The contribution

$$
\mathrm{P}\int_{-\infty}^{\infty} \frac{\lambda}{\xi - x}\left(\frac{A}{\pi}\right)\frac{1}{\xi^2 + A^2}\,\mathrm{d}\xi = -\frac{\lambda x}{x^2 + A^2}, \qquad (\mathrm{Im}\, x = 0),
\tag{26}
$$

5

of the density fluctuation to $i\dot{a}$ is real, and so tends to push $a(x)$ off the real axis. Its effect can be countered, however, by placing a solitary b-pole at $b = iA$. We then have

$$
\begin{aligned}
i\dot{a}(x,0) &= -\frac{\lambda x}{x^2 + A^2} + \frac{\lambda}{x - iA} \\
&= -\frac{\lambda x}{x^2 + A^2} + \frac{\lambda(x + iA)}{x^2 + A^2} \\
&= \frac{iA\lambda}{x^2 + A^2},
\end{aligned}
\tag{27}
$$

and the a-poles have a purely real initial velocity

$$
v(x,0) = \dot{a}(x,0) = \frac{A\lambda}{x^2 + A^2}.
\tag{28}
$$

They therefore stay on the real axis.

The motion of the $b = iA$-pole is obtained from

$$
\begin{aligned}
i\dot{b} &= -\int_{-\infty}^{\infty} \frac{\lambda}{b - \xi}\left(\frac{A}{\pi}\right)\frac{1}{\xi^2 + A^2}\,d\xi - \int_{-\infty}^{\infty}\frac{\lambda\rho_0}{b - \xi}\,d\xi \\
&= -\frac{\lambda}{b + iA} + i\lambda\pi\rho_0 \\
&= \frac{i\lambda}{2A} + i\lambda\pi\rho_0.
\end{aligned}
\tag{29}
$$

Thus, the b-pole velocity

$$
\dot{b} = \frac{\lambda}{2A} + \lambda\pi\rho_0
\tag{30}
$$

is also purely real. Since we know that the a-poles stay on the real axis, it must be that the a-pole density profile keeps abreast of the b-pole as it moves parallel to the real axis. A constant-shape solitary wave of density

$$
\rho(x,t) = \rho_0 + \left(\frac{A}{\pi}\right)\frac{1}{(x - v_{\text{soliton}}t)^2 + A^2}
\tag{31}
$$

therefore moves to the right at the speed $v_{\text{soliton}} = \dot{b}$.

We could have put the b-pole *below* the axis. In that case, both the a-poles and pulse envelope will move to the *left*. In either case, the envelope velocity is always higher than the speed of sound $v_{\text{sound}} = \lambda\pi\rho_0$, and is faster when the pulse envelope is tighter.

Observe how the b-pole acts as a *shepherd*: its real-part contribution serves to keep the a-poles from wandering off the real axis.

The a-poles are distributed in such a manner that the b-pole sees the effect of their enhanced density as a mirror image of itself lying below the axis. The contribution of this image to the b velocity parallels the interaction of a b and $a = b^*$ pair in the original real-valued Benjamin–Ono equation studied in [15]—the only difference being that in the present case we must include in $\dot{b}$ the constant velocity $v_{\text{sound}} = \lambda\pi\rho_0$ induced by the uniform ρ_0 background. The condition that $\dot{a}(0)$ be real is a linear equation linking the b-pole contribution to the density fluctuation $\delta\rho = (\rho - \rho_0)$. Consequently, the initial data for a chiral M-soliton can be established by placing poles above the axis at b_j, $j = 1, \ldots, M$. The associated $\delta\rho$ is then the sum of individual Lorentzians centred at $x_j = \text{Re}\,b_j$, and induces image poles below the axis at b_j^*, $j = 1, \ldots, M$. Thus, in our present approximation, the chiral Calogero hydrodynamics multisoliton coincides with the multisoliton solution of the conventional Benjamin–Ono equation. In particular, the real field

$$
\widetilde{u}(x,t) \overset{\text{def}}{=} \sum_{j=1}^{M}\left\{\frac{i\lambda}{x - b_j^*(t)} - \frac{i\lambda}{x - b_j(t)}\right\} + \lambda\pi\rho_0
\tag{32}
$$

obeys (1), and possesses the physical interpretations

$$\widetilde{u}(x, t) = v + \lambda\pi\rho = 2v + v_{\text{sound}} = \lambda\pi(2\rho - \rho_0). \tag{33}$$

This approximate mapping of the right-going Calogero density waves onto the Benjamin–Ono solitons is very appealing. Unfortunately, there is a fly in the ointment: the equation of continuity is not satisfied. For v and ρ both being functions of x and t in the combination $x - v_{\text{soliton}}t$, the particle-number conservation requires that

$$\dot{\rho} + \partial_x(\rho v) = \partial_x\{\rho(v - v_{\text{soliton}})\} = 0. \tag{34}$$

Because $\rho \to \rho_0$ and $v \to 0$ at a large distance, the right-hand side of (34) implies that

$$\rho(v_{\text{soliton}} - v) = \rho_0 v_{\text{soliton}} \tag{35}$$

or, equivalently,

$$v = \frac{\rho - \rho_0}{\rho} v_{\text{soliton}}. \tag{36}$$

Now the a-pole velocity and density that we have found are linked by $v(x) = \lambda\pi(\rho(x) - \rho_0)$, and since v_{soliton} is fixed while ρ in the numerator varies with the position, our present solution can obey (36) only approximately.

4. The nonlinear correction

To get the continuity equation to hold exactly, we need to improve on the crude approximation

$$\sum_{k;k\neq j} \frac{\lambda}{a_k - a_j} \to \text{P}\int_{-\infty}^{\infty} \rho(\xi)\frac{\lambda}{\xi - a_j}\,\mathrm{d}\xi. \tag{37}$$

This naïve continuum approximation would be legitimate (and (36) would hold exactly) were we to simultaneously let the background density ρ_0 become infinite and take $\lambda \to 0$ in such a way that $\rho_0\lambda$ remains constant. We are not doing this, however. We wish to keep λ fixed while ρ_0 becomes large, but remains finite. For finite λ, the poles immediately adjacent to a_j make a significant contribution to the sum, and their effect has to be carefully accounted for.

An improved approximation to the pole sum is

$$\sum_{k;k\neq j} \frac{\lambda}{a_k - a_j} \sim \text{P}\int_{-\infty}^{\infty} \frac{\lambda}{\xi - a_j}\rho(\xi, t)\,\mathrm{d}\xi - \frac{\lambda}{2}\partial_x \ln\rho(x)|_{x=a_j}. \tag{38}$$

The $\partial_x \ln\rho$ correction is the first term in an asymptotic series that expands the difference between the sum and integral in local gradients of ρ. It arises because the particle at a_j no longer lies midway between its neighbours at $a_{j\pm 1}$ when the density varies with position, but the symmetric cutoff in the principal-part integral tacitly assumes a midpoint location. We provide a derivation of this first correction term in the appendix.

Once we know of the local correction to the principal-part integral, we realize that the shepherd poles need to make a corresponding additional real contribution to i$\dot{a}$ if they are still to prevent the a_j-poles from wandering off the real axis. The equations determining this extra contribution are nonlinear, and so the simple pole superposition that enabled us to find the multisolition initial conditions in the previous section is no longer valid. We can still find some solutions, however. As an illustration, again consider the initial density profile

$$\rho(x, 0) = \rho_0 + \left(\frac{A}{\pi}\right)\frac{1}{x^2 + A^2}.$$

We now have

$$P\int_{-\infty}^{\infty} \frac{\lambda}{\xi - x}\rho(\xi, 0)\,d\xi - \frac{\lambda}{2}\partial_x \ln\rho = -\frac{\lambda x}{x^2 + A^2} - \frac{\lambda}{2}\left\{\frac{A}{\pi}\frac{-2x}{(x^2 + A^2)^2}\frac{1}{\rho_0 + \left(\frac{A}{\pi}\right)\frac{1}{x^2 + A^2}}\right\}$$

$$= -\frac{\lambda x}{x^2 + A^2 + A/\pi\rho_0}$$

$$= -\frac{\lambda x}{x^2 + B^2}, \tag{39}$$

where

$$B = \sqrt{A^2 + \frac{A}{\pi\rho_0}}. \tag{40}$$

We see that to keep a's on the real axis, the shepherd b-pole must be relocated to $b = iB$. The resultant a-pole motion is then governed by

$$i\dot{a}(x) = P\int_{-\infty}^{\infty}\frac{\lambda}{\xi - x}\rho(\xi, 0)\,d\xi - \frac{\lambda}{2}\partial_x \ln\rho + \frac{\lambda}{x - iB},$$

$$= -\frac{\lambda x}{x^2 + B^2} + \frac{\lambda}{x - iB},$$

$$= -\frac{\lambda x}{x^2 + B^2} + \frac{\lambda(x + iB)}{x^2 + B^2},$$

$$= \frac{iB\lambda}{x^2 + B^2}. \tag{41}$$

The new, improved, $v(x)$ therefore is

$$v(x) = \frac{\lambda B}{x^2 + B^2}. \tag{42}$$

From this, and (36), we get

$$v_{\text{soliton}} = \frac{\rho}{\rho - \rho_0}v(x)$$

$$= \rho_0\lambda\pi\left(\frac{B}{A}\right), \tag{43}$$

which is independent of x as it should be—and was *not* previously. By squaring this last equation, we also find that

$$A\pi\rho_0 = \frac{v_{\text{sound}}^2}{v_{\text{soliton}}^2 - v_{\text{sound}}^2}, \tag{44}$$

which is the relation between the soliton width and velocity obtained in [6].

Does this newly computed soliton velocity coincide with that of the shepherd b-pole? We set $b = iB$ in

$$i\dot{b} = -\frac{\lambda}{b + iA} + i\lambda\pi\rho_0 \tag{45}$$

to find

$$\dot{b} = \frac{\lambda}{A + B} + \lambda\rho_0\pi. \tag{46}$$

Multiplying both sides by $A(A + B)$ shows that $\dot{b} = v_{\text{soliton}}$, so the b-pole does indeed track the density pulse, as it must if the pulse is to retain its shape. This consistency check also illustrates the fact that the $\partial_x \ln\rho$ correction does not affect the contribution of the a-pole sum

away from the real axis. Once we are further away from the axis than the mean *a*-pole spacing, we no longer see the granularity of the *a*-pole 'charge' distribution, and so the naïve integral suffices for calculating the *a*-pole contribution to the *b*-pole motion.

We can also construct periodic solutions. If we take these to have period 2π, the resulting wave trains can be regarded as solitons of the trigonometric Sutherland model. Suppose, therefore, that the initial density profile is

$$
\begin{aligned}
\rho(x, 0) &= \rho_0 + \sum_{n=-\infty}^{\infty} \left(\frac{A}{\pi}\right) \frac{1}{(x + 2\pi n)^2 + A^2} \\
&= \rho_0 + \left(\frac{1}{2\pi}\right) \frac{\sinh A}{\cosh A - \cos x}.
\end{aligned}
\tag{47}
$$

Then

$$
\mathrm{P}\int_{-\infty}^{\infty} \frac{\lambda}{\xi - x} \rho(\xi, 0)\, \mathrm{d}\xi = \frac{-(\lambda/2)\sin x}{\cosh A - \cos x}
\tag{48}
$$

and

$$
\mathrm{P}\int_{-\infty}^{\infty} \frac{\lambda}{\xi - x} \rho(\xi, 0)\, \mathrm{d}\xi - \frac{\lambda}{2}\partial_x \ln \rho = \frac{-(\lambda/2)\sin x}{\cosh A - \cos x + \frac{\sinh A}{2\pi\rho_0}}.
\tag{49}
$$

This real contribution to i$\dot{a}$ can be cancelled by placing an infinite train of shepherd poles at $z_n = iB + 2\pi n$. From

$$
\lim_{M\to\infty}\left\{ \sum_{n=-M}^{M} \frac{1}{(x - iB) + 2\pi n} \right\} = \frac{1}{2}\cot\left(\frac{x - iB}{2}\right) = \frac{1}{2}\left(\frac{\sin x + i\sinh B}{\cosh B - \cos x}\right),
\tag{50}
$$

we see that B should be chosen so that

$$
\cosh B = \cosh A + \frac{1}{2\pi\rho_0}\sinh A.
\tag{51}
$$

By algebra that parallels the single-soliton case, we find that

$$
v(x, 0) = \left(\frac{\lambda}{2}\right)\frac{\sinh B}{\cosh B - \cos x},
\tag{52}
$$

and that the soliton velocity is given by

$$
v_{\mathrm{soliton}} = \frac{\rho}{\rho - \rho_0}v(x) = \rho_0\lambda\pi\left(\frac{\sinh B}{\sinh A}\right).
\tag{53}
$$

For a wave train of period Λ, these equations become

$$
v(x, 0) = \left(\frac{\lambda\pi}{\Lambda}\right)\frac{\sinh(2\pi B/\Lambda)}{\cosh(2\pi B/\Lambda) - \cos(2\pi x/\Lambda)},
\tag{54}
$$

and

$$
v_{\mathrm{soliton}} = \rho_0\lambda\pi\frac{\sinh(2\pi B/\Lambda)}{\sinh(2\pi A/\Lambda)}.
\tag{55}
$$

They reduce to (42) and (43), respectively, when Λ becomes large.

5. The general case

We now show that the local correction guarantees consistency with particle conservation for arbitrary initial data.

To motivate the discussion, we begin by considering a simple model for the dynamics of a one-dimensional gas of spinless, unit-mass, fermions. The uniform non-interacting gas would have internal energy density $\epsilon(\rho) = \hbar^2 \pi^2 \rho^3 / 6$. To include the effect of some interactions, we generalize this expression by introducing a parameter λ so that $\epsilon(\rho) = \lambda^2 \pi^2 \rho^3 / 6$. This λ will later be identified with the parameter λ appearing in the earlier sections. Then the simplest classical Galiliean-invariant Hamiltonian for the gas is

$$H_{\text{fluid}} = \int \left\{ \frac{1}{2} \rho v^2 + \frac{\lambda^2 \pi^2}{6} \rho^3 \right\} \, \mathrm{d}x. \tag{56}$$

Here, v is the fluid velocity. We can write $v = \partial_x \theta$, where $\theta(x, t)$ is a phase field canonically conjugate to the density. The field θ has Poisson bracket

$$\{\theta(x), \rho(x')\} = \delta(x - x), \tag{57}$$

leading to

$$\{v(x), \rho(x')\} = \partial_x \delta(x - x'), \qquad \{v(x), v(x')\} = \{\rho(x), \rho(x')\} = 0. \tag{58}$$

From $\dot{\rho} = \{H_{\text{fluid}}, \rho\}$, we obtain the equation of continuity

$$\dot{\rho} + \partial_x (\rho v) = 0, \tag{59}$$

and from $\dot{v} = \{H_{\text{fluid}}, v\}$, we obtain Euler's equation

$$\dot{v} + v \partial_x v = -\partial_x \left(\frac{\lambda^2 \pi^2 \rho^2}{2} \right). \tag{60}$$

We can rearrange H_{fluid} as

$$H_{\text{fluid}} = \frac{1}{2\lambda\pi} \int \left\{ \frac{1}{6} (v + \lambda\pi\rho)^3 - \frac{1}{6} (v - \lambda\pi\rho)^3 \right\} \, \mathrm{d}x, \tag{61}$$

and the equations of motion can similarly be massaged into the Riemann form as

$$\begin{aligned} \partial_t (v + \lambda\pi\rho) + (v + \lambda\pi\rho) \partial_x (v + \lambda\pi\rho) &= 0, \\ \partial_t (v - \lambda\pi\rho) + (v - \lambda\pi\rho) \partial_x (v - \lambda\pi\rho) &= 0. \end{aligned} \tag{62}$$

We therefore have non-communicating right-going and left-going Riemann invariants $I_{\text{R,L}}(x) = (v \pm \lambda\pi\rho)$ that are proportional to the chiral currents $j_{\text{R,L}} = \frac{1}{2} (\rho \pm v/\lambda\pi)$ associated with the right and left Fermi points respectively. The Riemann invariants have Poisson brackets

$$\begin{aligned} \{I_{\text{R}}(x), I_{\text{R}}(x')\} &= -2\lambda\pi \, \partial_x \delta(x - x'), \\ \{I_{\text{L}}(x), I_{\text{L}}(x')\} &= -2\lambda\pi \, \partial_x \delta(x - x'), \\ \{I_{\text{R}}(x), I_{\text{L}}(x')\} &= -0. \end{aligned} \tag{63}$$

Riemann's equations also show that this simple model contains the seeds of its own destruction—the leading edge of any simple wave will inevitably steepen and break, making the fields unphysically multivalued.

We now show how the Calogero gas mimics the Fermi gas, while regularizing the multivaluedness.

Following [9] we decompose the u field in (5) as $u(z, t) = u_+(z, t) + u_-(z, t)$, where

$$u_-(z, t) = \sum_{j=1}^{N} \frac{i\lambda}{z - a_j(t)}, \tag{64}$$

$$u_+(z, t) = \sum_{j=1}^{M} \frac{-i\lambda}{z - b_j(t)}. \tag{65}$$

$u_\pm(z, t)$ are eigenfunctions of the Γ-contour Hilbert transform with eigenvalues $\pm i$ respectively. In the hydrodynamic limit, u_- becomes

$$u_-(z, t) = i \int_{-\infty}^{\infty} \rho(\xi, t) \frac{\lambda}{z - \xi} \, d\xi, \tag{66}$$

and has a discontinuity across the real axis:

$$u_-(x + i\epsilon) - u_-(x - i\epsilon) = 2\lambda\pi\rho(x). \tag{67}$$

Thus,

$$u_-(x \pm i\epsilon) = \lambda\pi(i\rho_H \pm \rho). \tag{68}$$

With the local correction included, the a-pole velocity $v = \dot{a}$ is

$$v = i\lambda\pi\rho_H + \frac{i\lambda}{2} \partial_x \ln \rho + u_+. \tag{69}$$

We can rearrange the last equation as

$$u_+(x, t) = v - i\lambda\pi\rho_H - \frac{i\lambda}{2} \partial_x \ln \rho. \tag{70}$$

For general $\rho(x)$, $v(x)$, we must take (70) to be the *definition* of $u_+(x)$ on the real axis. We then define $u_+(z)$ to be the analytic continuation of $u_+(x)$ away from the axis. The analytically continued u_+ will have no singularities within Γ, but will only be of the simple form (65) for the restricted set of initial data that leads to pure soliton solutions. Nonetheless, as we show in the appendix, the only requirement for a-pole autonomy is that $(u_+)_\Gamma = iu_+$.

The total $u = u_+ + u_-$ field therefore has limits above and below the real axis equal to

$$u(x \pm i\epsilon) = v \pm \lambda\pi\rho - \frac{i\lambda}{2} \partial_x \ln \rho. \tag{71}$$

These boundary values $u(x \pm i\epsilon)$ almost coincide with the Riemann invariants $I_{R,L}(x)$. The only difference is a shift $v \to v - i\lambda\partial_x \ln \rho/2$, and this shift does not affect the Poisson algebra in (63). In the quantum theory, the invariance comes about because the shift is effected by a conjugation [13], and conjugation does not alter c-number commutators. In the classical theory, we establish the invariance by observing that

$$\left\{ \theta(x) - i\lambda\tfrac{1}{2} \ln \rho(x), \theta(x') - i\lambda\tfrac{1}{2} \ln \rho(x') \right\} = 0, \tag{72}$$

and so, on differentiating by x and x', we find

$$\left\{ v(x) - i\lambda\tfrac{1}{2} \partial_x \ln \rho(x), v(x') - i\lambda\tfrac{1}{2} \partial_{x'} \ln \rho(x') \right\} = 0. \tag{73}$$

The shifted velocity therefore still Poisson commutes with itself. Showing the invariance of the other brackets is straightforward. In particular, we find that $u(x + i\epsilon)$ has vanishing Poisson bracket with $u(x - i\epsilon)$. Thus, the real-axis fluid-dynamics Poisson algebra (58) is equivalent to the natural extension of the Benjamin–Ono Poisson algebra (3) to the contour Γ.

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We can also reduce the Γ-contour version of the Benjamin–Ono Hamiltonian

$$H_\Gamma = \frac{1}{2\lambda\pi} \oint_\Gamma \left\{ \frac{1}{6} u^3 - \frac{\lambda}{4} u(\partial_z u)_\Gamma \right\} dz \tag{74}$$

to a real-axis integral. Using expression (71) for $u(x \pm i\epsilon)$ and $u_\Gamma = i(u_+ - u_-)$ we find, after some labour, that this integral is

$$H_{\text{hydro}} = \int_{-\infty}^\infty \left\{ \frac{1}{2}\rho v^2 + \frac{\lambda^2 \pi^2}{6}\rho^3 - \frac{\lambda^2 \pi}{2}\rho(\partial_x \rho)_H + \frac{\lambda^2}{8}\frac{(\partial_x \rho)^2}{\rho} \right\} dx. \tag{75}$$

From the general theory, we know that (74) leads to the Benjamin–Ono equation on Γ, and that, in turn, leads to the Calogero equation on the real axis. Thus H_{hydro}, when used in conjunction with the real-axis fluid-dynamics Poisson algebra (58), must be the Hamiltonian governing the real-axis hydrodynamics. It therefore follows, from $\dot\rho = \{H_{\text{hydro}}, \rho\} = -\partial_x(\rho v)$, that the equation of continuity is exactly satisfied.

H_{hydro} is the continuum Hamiltonian of [6, 7]. In those works, it was obtained indirectly by taking the $\hbar \to 0$ limit of the quantum continuum Hamiltonian that was already known from the collective field approach to matrix models [16]. In the appendix we provide a more direct, and purely classical, derivation of the rather less-than-obvious expression for the internal energy part of H_{hydro}:

$$V[\rho] = \frac{1}{2}\sum_{i \neq j} \frac{\lambda^2}{(a_i - a_j)^2} \sim \int_{-\infty}^\infty \left\{ \frac{\lambda^2 \pi^2}{6}\rho^3 - \frac{\lambda^2 \pi}{2}\rho(\partial_x \rho)_H + \frac{\lambda^2}{8}\frac{(\partial_x \rho)^2}{\rho} \right\} dx. \tag{76}$$

Although the boundary values $u(x \pm i\epsilon)$ of the $u(z, t)$ field Poisson commute, the left- and right-going Riemann invariants are no longer dynamically decoupled. For example, an interplay between the two segments of the contour Γ is clearly seen when we use the u-field boundary values to compute the Γ-contour Hilbert transform of u at $z = x - i\epsilon$:

$$
\begin{aligned}
u_\Gamma(x - i\epsilon) &= \frac{P}{\pi} \oint_\Gamma \frac{u(z')}{(x - i\epsilon) - z'} dz' \\
&= \frac{1}{\pi} \int_{-\infty}^\infty \frac{u(x' + i\epsilon)}{(x - i\epsilon) - (x' + i\epsilon)} dx' - \frac{P}{\pi} \int_{-\infty}^\infty \frac{u(x' - i\epsilon)}{x - x'} dx' \\
&= \frac{P}{\pi} \int_{-\infty}^\infty \frac{u(x' + i\epsilon) - u(x - i\epsilon)}{x - x'} dx' + \frac{1}{\pi}\{i\pi u(x + i)\} \\
&= \frac{P}{\pi} \int_{-\infty}^\infty \frac{2\pi\rho(x')}{x - x'} dx' + iu(x + i\epsilon), \\
&= 2\pi\rho_H + i\left(v + \lambda\pi\rho - i\lambda\frac{1}{2}\partial_x \ln\rho\right) \\
&= i(u_+ - u_-)|_{z=x-i\epsilon}.
\end{aligned}
\tag{77}
$$

In the second line, the two integrals are the contributions from the upper and lower segments of Γ respectively. $u(x + i\epsilon)$ appearing in the third line is the delta-function contribution from

$$\frac{1}{x - x' - i\epsilon} = P\left(\frac{1}{x - x'}\right) + i\pi\delta(x - x') \tag{78}$$

in the upper-segment integration.

The interaction between the two branches of Γ also shows up when we seek chiral (i.e. unidirectional) waves. To have purely right-going motion, all the $u_+(z)$ singularities must lie in the upper half-plane. From the theory of Hilbert transforms, this condition is equivalent to demanding that

$$(u_+)_H = iu_+. \tag{79}$$

We can arrange for this eigenvalue condition to hold by imposing the chiral constraint [9, 12, 13]

$$v = \lambda\pi(\rho - \rho_0) - \tfrac{1}{2}\lambda(\partial_x \ln \rho)_H, \qquad (80)$$

whence

$$u_+ = \lambda\pi[(\rho - \rho_0) - i\rho_H] - \frac{i}{2}\lambda[\partial_x \ln \rho - i(\partial_x \ln \rho)_H]. \qquad (81)$$

This last form of u_+ obeys (79) because $(f_H)_H = -f$ for functions in $L^2(\mathbb{R})$. (Note that square integrability requires us to subtract the background density from ρ in these equations. The Hilbert transform of a constant is zero, and $(\rho_H)_H = -(\rho - \rho_0)$.) Under the chiral condition, the equation of continuity and the Euler equation reduce to a single equation

$$\dot{\rho} + \partial_x[\lambda\pi(\rho - \rho_0)\rho] = \frac{\lambda}{2}\partial_x[\rho(\partial_x \ln \rho)_H], \qquad (82)$$

This strongly nonlinear equation reduces to the right-going Benjamin–Ono equation for the $\widetilde{u} = \lambda\pi(2\rho - \rho_0)$ field introduced in equations (32) and (33) if we linearize the dispersive term $\rho(\partial_x \ln \rho)_H \to (\partial_x \rho)_H$, but in doing this we abandon strict number conservation.

6. Discussion

We have seen how the Benjamin–Ono equation on the 'double' naturally generates the solitary-wave and wave-train solutions in the classical hydrodynamic limit of the Calogero–Sutherland model. We have also shown how this approach leads to an understanding of the origin of the subtle corrections to the naïve continuous-fluid limit. We could have computed these corrections to higher order in the gradients of ρ. The small parameter in this expansion is, however, the local change in the inter-particle spacing divided by the spacing itself and should be small in the hydrodynamic limit. Keeping track of these higher order terms would undo the advantages of the continuous-fluid approximation. We therefore retain only those that are required to maintain the internal consistency of the fluid mechanics model.

It is interesting to compare the manner in which the classical machinery works with the quantum mechanical formalism of [9, 13]. To make this comparison, we should first note that the parameter λ that appears in the quantum model is a dimensionless number. Our present classical parameter λ has a scale, and is related to the quantum parameter by

$$\lambda_{\text{classical}} = \hbar\lambda_{\text{quantum}}. \qquad (83)$$

Thus, the free-fermion limit, where $\lambda_{\text{quantum}} = 1$, corresponds to $\lambda_{\text{classical}} = \hbar$, and any distinction between $\lambda_{\text{quantum}} = 1$ and $\lambda_{\text{quantum}} = 0$ is invisible in the classical $\hbar \to 0$ picture. What can be seen in the classical picture is the analytic structure of the $u(z, t)$ field and the resulting interplay of the left- and right-going currents $I_{\text{R,L}}(x) \simeq u(x \pm i\epsilon)$ with the positive and negative parts $u_\pm(x)$ of the mode expansion. In particular, we see how these ingredients combine in a Poisson-bracket algebra that is the classical version of the quantum mechanical current algebra.

Acknowledgments

Appendix

In this appendix, we provide proofs of some assertions made in the main text.

A.1. Singular sums

Here we obtain the continuous-fluid approximations to the sums

$$S_1 = \sum_{n=-\infty}^{\infty} \frac{1}{a_n - a_0}, \tag{A.1}$$

$$S_2 = \frac{1}{2} \sum_{n,m=-\infty}^{\infty} \frac{1}{(a_m - a_n)^2}. \tag{A.2}$$

As usual, the singular $a_n = a_0$ and $a_n = a_m$ terms are to be omitted in their respective sums. The essential tool is the Euler–Maclaurin expansion:

$$\sum_{n=1}^{\infty} f(n) \sim \int_0^{\infty} f(\nu)\, d\nu - \frac{1}{2} f(0) - \frac{1}{12} f'(0) + \frac{1}{720} f'''(0) + \cdots. \tag{A.3}$$

This asymptotic expansion is valid for smooth functions $f(\nu)$. The sums we are evaluating require singular f's, however, and so a strategy of subtractions is needed.

We regard the discrete numbers a_n as the values at $\nu = n \in \mathbb{Z}$ of a smooth monotonic-increasing function $a(\nu)$. We wish to form the sum

$$S_1 = \lim_{N \to \infty} \left\{ \sum_{n=-N}^{N} \frac{1}{a(n) - a(0)} \right\}, \tag{A.4}$$

where the term with $n = 0$ is to be omitted. We replace this sum by

$$S_1' = \lim_{N \to \infty} \sum_{n=-N}^{N} \left\{ \frac{1}{a(n) - a(0)} - \frac{1}{a'(0)n} \right\}, \tag{A.5}$$

where again the term with $n = 0$ is to be omitted. Now $S_1' = S_1$ because the subtracted sum is zero—all its terms cancel in pairs. Thus,

$$S_1 = \lim_{N \to \infty} \left\{ \sum_{n=1}^{N} [f(n) + f(-n)] \right\}, \tag{A.6}$$

where

$$f(\nu) = \frac{1}{a(\nu) - a(0)} - \frac{1}{a'(0)\nu} \tag{A.7}$$

has a smooth $\nu \to 0$ limit:

$$\lim_{\nu \to 0} f(\nu) = -\frac{1}{2} \frac{a''(0)}{[a'(0)]^2}. \tag{A.8}$$

Keeping only the first correction in the Euler–Maclaurin series, we have

$$\int_0^{\infty} f(\nu)\, d\nu \sim \frac{1}{2} f(0) + f(1) + f(2) + \cdots,$$
$$\int_{-\infty}^{0} f(\nu)\, d\nu \sim \frac{1}{2} f(0) + f(-1) + f(-2) + \cdots, \tag{A.9}$$

and so

$$S_1 \sim \int_{-\infty}^{\infty} f(\nu)\, d\nu + \frac{1}{2} \frac{a''(0)}{[a'(0)]^2} + \cdots. \tag{A.10}$$

The integral in this last expression is convergent at $\nu = 0$ since $f(\nu)$ is finite there. Because the Hilbert transform of a constant vanishes, we can, however, remove the counter-term that makes

$f(0)$ finite provided we cut off the $\nu \to 0$ divergence with the principal-part prescription. On doing this, we obtain

$$
\begin{aligned}
S_1 &\sim \mathrm{P}\int_{-\infty}^{\infty} \frac{1}{a(\nu) - a(0)}\, \mathrm{d}\nu + \frac{1}{2}\frac{a''(0)}{[a'(0)]^2} \\
&= \mathrm{P}\int_{-\infty}^{\infty} \frac{\rho(\xi)}{\xi - a_0}\, \mathrm{d}\xi - \frac{1}{2}\partial_x \ln \rho(x)|_{x=a_0}.
\end{aligned}
\tag{A.11}
$$

In passing from the first line to the second, we have made a smooth change of variables $\nu \mapsto \xi = a(\nu)$ (such changes of variables are legitimate in principal-part integrals) and defined the particle density $\rho(x)$ in terms of $a(\nu)$ by setting

$$
a'(\nu) = \frac{\mathrm{d}a}{\mathrm{d}\nu} = \frac{\mathrm{d}x}{\mathrm{d}\nu} = \frac{1}{\rho(x)}.
\tag{A.12}
$$

We have also used

$$
a''(\nu) = \frac{\mathrm{d}^2 a}{\mathrm{d}\nu^2} = -\frac{1}{\rho^2}\frac{\mathrm{d}\rho}{\mathrm{d}\nu} = -\frac{1}{\rho^3}\frac{\mathrm{d}\rho}{\mathrm{d}x}.
\tag{A.13}
$$

To evaluate the double sum S_2, we begin by obtaining the continuum approximation to

$$
S_3 = \sum_{n=-\infty}^{\infty} \frac{1}{(a(n) - a(0))^2},
\tag{A.14}
$$

where the $n = 0$ term is to be omitted. Following our subtraction strategy, we use Euler's formula $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$ to write

$$
S_3 - \frac{1}{[a'(0)]^2}\frac{\pi^2}{3} = \lim_{N\to\infty}\left\{ \sum_{n=1}^{N}[F(n) + F(-n)] \right\},
\tag{A.15}
$$

where

$$
F(\nu) = \frac{1}{(a(\nu) - a(0))^2} - \frac{1}{[a'(0)]^2}\frac{1}{\nu^2} + \frac{a''(0)}{[a'(0)]^3}\frac{1}{\nu}
\tag{A.16}
$$

has been constructed to have a finite $\nu \to 0$ limit:

$$
\lim_{\nu\to 0} F(\nu) = -\frac{1}{3}\frac{a'''(0)}{[a'(0)]^3} + \frac{3}{4}\frac{[a''(0]^2}{[a'(0)]^4}.
\tag{A.17}
$$

We apply the Euler–Maclaurin formula as before to find that

$$
S_3 \sim \frac{\pi^2}{3}\frac{1}{[a'(0)]^2} + \int_{-\infty}^{\infty} F(\nu)\, \mathrm{d}\nu - F(0) + \cdots.
\tag{A.18}
$$

We can again omit the $1/\nu$ term in $F(\nu)$ at the expense of replacing the convergent integral by a principal-part integral.

We now claim that

$$
\mathrm{P}\int_{-\infty}^{\infty}\left\{ \frac{1}{(a(\nu) - a(0))^2} - \frac{1}{[a'(0)]^2}\frac{1}{\nu^2} \right\} \mathrm{d}\nu = \mathrm{P}\int_{-\infty}^{\infty} \frac{(\rho(x) - \rho(0))}{(x - a_0)^2}\, \mathrm{d}x.
\tag{A.19}
$$

This result may be obtained by making a substitution $\nu \mapsto x = a(\nu)$ in the first term of the integrand on the left-hand side and a substitution $\nu \mapsto x = a(0) + a'(0)\nu$ in the second term. Now it is not immediately obvious that the use of two distinct changes of variable is legitimate. The integrals of the two terms do not exist separately—only the integral of their difference is convergent at $\nu = 0$. In order to be sure that no additional finite contribution is introduced by our manoeuvre, we must provide a common $|\nu| > \epsilon$ cutoff for the two separate integrals, and then keep a track of the effect of the subsequent changes of variables on their integration

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limits. Because the two changes of variables agree to linear order near $v = 0$, we find that no such finite additions are induced. Our manoeuvre is indeed allowed. We also note that

$$\mathrm{P} \int_{-\infty}^{\infty} \frac{(\rho(\xi) - \rho(0))}{(\xi - x)^2} \mathrm{d}\xi = \frac{\mathrm{d}}{\mathrm{d}x} \left(\mathrm{P} \int_{-\infty}^{\infty} \frac{\rho(\xi)}{\xi - x} \mathrm{d}\xi \right) = -\pi \partial_x (\rho(x)_{\mathrm{H}}) = -\pi (\partial_x \rho)_{\mathrm{H}}. \quad \text{(A.20)}$$

Next, we re-express $F(0)$ as

$$F(0) = -\frac{1}{4} \frac{[a''(0)]^2}{[a'(0)]^4} - \frac{1}{3} \frac{\mathrm{d}}{\mathrm{d}v} \left(\frac{a''(v)}{[a'(v)]^3} \right) \bigg|_{v=0}. \quad \text{(A.21)}$$

To complete the evaluation of S_2, we should replace a_0 by a_n and sum over n. In the continuum approximation, we replace $a(0)$ by $a(v)$ and integrate over v. The total derivative in F will not contribute to this integral and can be discarded. After changing variables $v \to x$, we therefore find that

$$S_2 \sim \int_{-\infty}^{\infty} \left\{ \frac{\pi^2}{6} \rho^3 - \frac{\pi}{2} \rho (\partial_x \rho)_{\mathrm{H}} + \frac{1}{8} \frac{(\partial_x \rho)^2}{\rho} \right\} \mathrm{d}x + \cdots, \quad \text{(A.22)}$$

which immediately gives equation (76).

A.2. Pole autonomy for a general u_+

We here show that we can relax the condition that u_+ be a sum of simple poles, yet still have a_j obey the Calogero equation (11).

Let

$$u_-(z, t) = \sum_{j=1}^{N} \frac{\mathrm{i}\lambda}{z - a_j(t)}, \quad \text{(A.23)}$$

as before, but assume of $u_+(z, t)$ only that $(u_+)_{\Gamma} = +\mathrm{i}u_+$. Insert $u = u_- + u_+$ into

$$\dot{u} + u \partial_z u = \tfrac{1}{2} \partial_{zz}^2 u_{\Gamma}. \quad \text{(A.24)}$$

The projections of the product $u_+ u_-$ onto the $\pm \mathrm{i}$ eigenspaces of the Hilbert transform are respectively

$$(u_+ u_-)_+ = u_+ u_- - \sum_{j=1}^{N} \frac{\mathrm{i}\lambda}{z - a_j} u_+(a_j),$$

$$(u_+ u_-)_- = \sum_{j=1}^{N} \frac{\mathrm{i}\lambda}{z - a_j} u_+(a_j).$$

By taking the z derivative of (A.25), we can project the cross terms $u_+ \partial_z u_- + u_- \partial_z u_+$ appearing in (A.24) into the $\pm \mathrm{i}$ eigenspaces . From the coefficients of $1/(z - a_j)^2$ in the $-\mathrm{i}$ eigenspace, we read off from (A.24) that

$$\mathrm{i}\dot{a}_j = \sum_{k; k \neq j} \frac{\lambda}{a_k - a_j} + \mathrm{i}u_+(a_j). \quad \text{(A.26)}$$

In the $+\mathrm{i}$ eigenspace, we find that (A.24) requires that

$$\dot{u}_+ + u_+ \partial_z u_+ + \sum_{k=1}^{N} \frac{\mathrm{i}\lambda}{z - a_k} + \sum_{k=1}^{N} \frac{\mathrm{i}\lambda}{(z - a_k)^2} (u_+(a_k) - u_+(z)) = \mathrm{i}\lambda \frac{1}{2} \partial_{zz}^2 u_+. \quad \text{(A.27)}$$

Now again differentiate $\dot{a}_j$ to find that

$$\mathrm{i}\ddot{a}_j = \sum_{k; k \neq j} \frac{\lambda}{(a_k - a_j)^2} (\dot{a}_j - \dot{a}_k) + i(\dot{u}_+(a_j) + \dot{a}_j \partial_z u_+|_{z=a_j}). \quad \text{(A.28)}$$

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If we momentarily forget all about u_+, we know that we can assemble the remaining terms to obtain the Calogero equation

$$\ddot{a}_j = \sum_{k;k\neq j} \frac{2\lambda^2}{(a_j - a_k)^3}.$$

We therefore need to show that all terms in $\mathrm{i}\ddot{a}_j$ involving u_+ drop out. These terms are

$$\sum_{k;k\neq j} \frac{\lambda}{(a_k - a_j)^2}(u_+(a_j) - u_+(a_k)) + \mathrm{i}\left(\dot{u}_+(a_j) + \sum_{k;k\neq j} \frac{\mathrm{i}\lambda}{a_j - a_k}\partial_z u_+|_{a_j} + u_+\partial_z u_+|_{a_j}\right). \tag{A.29}$$

Now consider the limit of (A.27) as $z \to a_j$. There are potential singularities arising from the $k = j$ terms in the sums, but on expanding

$$u_+(z) = u_+(a_j) + (z - a_j)\partial_z u_+|_{a_j} + \tfrac{1}{2}(z - a_j)^2\partial_{zz}^2 u_+|_{a_j} + O[(z - a_j)^3],$$

we find that the potentially singular parts cancel among themselves, and, furthermore, the remaining finite parts of the $j = k$ terms combine to cancel the $\mathrm{i}\lambda\frac{1}{2}\partial_{zz}^2 u_+$ term on the right-hand side. The $z = a_j$ limit of (A.27) thus equates to zero precisely the terms (A.29) that we wish to disappear.

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